The charge radii and the decay rates of the pseudoscalar meson

Yongkyu Ko *

Department of Physics, Yonsei University, Seoul 120-749, Korea

(August 18, 1999)

The charge radii and the decay rates of the pion and kaons are calculated, using the relativistic equation of motion with a linear potential. Those physical quantities are quite well explained with the current quark masses in the case of the pion. It is found that the Van Royen-Weisskopf paradox can be cleared out only in the linear potential model by considering the color degree of freedom of the quark in the meson. The physical quantities for the kaon are not so satisfactory that it should be required to reconsider the Cabibbo-Kobayashi-Maskawa matrix element in the kaon decay.

PACS numbers: 12.39.Pn, 12.39.Ki, 13.20.-v, 21.10.Ft

Recently the author has attempted to derive a quark confinement potential from quantum chromodynamics (QCD) [1]. Unfortunately the attempt does not yet succeed, because there is a serious problem to define the Fourier transform of a confinement potential which is non-local. However it is natural to think that there should exist a certain confinement potential in nature, considering hadronic phenomena. So there are some attempts to define it by introducing a small parameter [2,3]. This letter has the purpose to explain some physical quantities using the potential in order to confirm the existence of such a confinement potential in nature. The electromagnetic charge radii and the decay rates of the pseudoscalar meson are very important physical quantities for understanding the structure of the hadron and QCD of its theory. Those physical quantities have been calculated many times since the advent of the quark model [4–8]. However there are some difficulties to describe hadronic phenomena, such as the constituent quark masses and the Van Royen-Weisskopf paradox in those literatures. This letter shows that the proper relativistic description explains such difficulties quite reasonably with the current quark masses. The consideration of the color degree of freedom clears out the paradox excellently, which is the difference of the spatial wave function $|\Psi(0)|^2$ between the calculations of the charge radius and the decay rate of the pion. In the case of the kaon, however, there is a difficulty to explain its charge radii and decay rate consistently. The reason is conjectured that the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{us}|^2$ is too large so that the kaon decay constant is too small. Therefore it is necessary to scrutinize the CKM matrix element for the kaon decay.

In order to describe the relativistic feature of the quark in the bound system of the meson, the Klein Gordon equation is used to do it rather than the Dirac equation. Since the quark is, of course, a fermion, the Dirac equation should be used to do it. However there are good reasons to adopt the Klein Gordon equation, because the degeneracy of spin singlet and triplet states is broken larger than the masses of pseudoscalar mesons so that the spin structure of the equation of motion may be less important. Moreover the relative motion of the quark in the meson can be assumed to be governed by the effective potential of the spin singlet state. While the electron in the hydrogen atom has two spin states to occupy in the ground state regardless of the spin of the proton, the quark has only one spin state in the ground state of the meson due to the influence of the other quark, because they have similar masses. This is another reason to adopt the Klein Gordon equation, because it is not necessary to describe the motion of the quark with a spinor.

The Klein Gordon equation for a free quark is given by

$$(\Box + m^2)\Psi = 0. \tag{1}$$

Since two body system can be separated into the relative and the center of mass parts, if the equation is modified as follows

$$\begin{aligned} -\frac{1}{m_1}\nabla_1^2\Psi - (\frac{E_1^2}{m_1} - m_1)\Psi &= 0 \\ -\frac{1}{m_2}\nabla_2^2\Psi - (\frac{E_2^2}{m_2} - m_2)\Psi &= 0. \end{aligned} \tag{2}$$

*Electronic address: yongkyu@phya.yonsei.ac.kr

Using the identity

$$\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2 = \frac{1}{\mu}\nabla_r^2 + \frac{1}{m_1+m_2}\nabla_R^2, \tag{3}$$

where μ is the reduce mass of m_1 and m_2, the sum of the two equations is rewritten as

$$-\frac{1}{\mu}\nabla_r^2\Psi - \frac{1}{m_1+m_2}\nabla_R^2\Psi - (\frac{\boldsymbol{p}_1^2}{m_1} + \frac{\boldsymbol{p}_2^2}{m_2})\Psi = 0. \tag{4}$$

At the center of mass frame, $\nabla_R^2\Psi = 0$ and $\boldsymbol{p}_1^2 = \boldsymbol{p}_2^2 = \boldsymbol{p}^2$, the equation is written simply as

$$-\frac{1}{\mu}\nabla_r^2\Psi - \frac{\boldsymbol{p}^2}{\mu}\Psi = 0, \tag{5}$$

where $\boldsymbol{p}^2 = E_1^2 - m_1^2 = E_2^2 - m_2^2 = E_r^2 - \mu^2 = \boldsymbol{p}_r^2$. The relative energy-momentum and mass are $E_r = m_2E_1'/(m_1 + m_2), \boldsymbol{p}_r = m_2\boldsymbol{p}_1'/(m_1+m_2)$, and $\mu = m_2m_1/(m_1+m_2)$, where E_1' and $\boldsymbol{p}_1'$ are the energy and momentum in the rest frame of m_2. Thus the relative part of the two body system is reduced to the equation of motion of a single particle:

$$(\Box + \mu^2)\Psi = 0, \tag{6}$$

which satisfies the relative energy-momentum relation. Since the two particles interact with each other with exchanging their gauge field, the derivative should be replaced to the covariant derivative as follows

$$(D_\mu D^\mu + \mu^2)\Psi = 0. \tag{7}$$

where $D_\mu = \partial_\mu - igA_\mu$. A few algebraic effort can make the equation be decomposed as

$$-\frac{1}{E_r+\mu}\nabla^2\Psi + \frac{1}{E_r+\mu}(2gA_0E_r - 2g\boldsymbol{A}\cdot\boldsymbol{p}_r - g^2A_0^2 + g^2\boldsymbol{A}^2)\Psi = (E_r-\mu)\Psi, \tag{8}$$

where the color index is hidden in the case of QCD. For the test of the validity of the equation, let's approximate it for the case of QED $E \approx \mu$, namely, $E_r = \mu + K_r$ and $K_r \ll \mu$. Thus the equation is expanded as

$$-\frac{1}{2\mu}\nabla^2\Psi + eA_0\Psi - \frac{e}{\mu}\boldsymbol{A}\cdot\boldsymbol{p}_r\Psi + \frac{e^2}{2\mu}\boldsymbol{A}^2\Psi - \frac{1}{2\mu}[(eA_0)^2 - (K_reA_0) + \{K_r(-\frac{\nabla^2}{2\mu})\}]\Psi = K_r\Psi, \tag{9}$$

where higher order terms are ignored. If the potential $A_0 = -e/r$ is inserted, this equation is nothing but the Schrödiger equation with the relativistic correction terms in the square bracket which agree to that in the standard texts by inserting the relation $K_r = -\nabla^2/2\mu + eA_0$ in the brace bracket. In the nonrelativistic case of the motion of the particle, the potential which make a bound system is the coulomb one. However the potential should be modified as the effective potential as insisted in Ref [1] due to the effect of the vacuum polarization of the quark, when the motion of the particle reaches to the ultra relativistic region. Hence Eq. (8) can be written as

$$-\nabla^2\Psi + 2E_r(V_{\text{eff.}} - E_r^2 + \mu^2)\Psi = 0. \tag{10}$$

Notice the coefficient $2E_r$ in front of the effective potential which is 2μ in the nonrelativistic case. It seems to be the origin to introduce the constituent quark masses in the calculations of low energy physical quantities with the Schrödinger equation and other relativistic approaches. For the confinement potential $V_{\text{eff.}} = kr$, where $k = 2\alpha_s^2/3\pi\Sigma_f m_f^2$ under the condition $4m_f^2 < q_{\text{gluon}}^2$ in Ref. [1], the solution of the differential equation is just the Airy function which is

$$\Psi(r) = Cf(r)e^{-ar^{3/2}}, \tag{11}$$

where $C = \sqrt{3a^2/2\pi}$ for $f(r) = 1$ which gives the finite value $|\Psi(0)|^2$ in the calculations of decay rates. The parameter a is calculated as $\sqrt{8E_rk/9}$ from the differential equation for the ground state of the quark. Since the coefficient of the wave function depends on the energy of the particle instead of its mass, it is natural that the minima for the s- and d-wave components of the deuteron function should be shifted toward large momenta in relativistic case as pointed out in Ref. [9]. In this reference, the potential should be also transformed or rewritten by a scale transformation in the momenta, but such a strange thing does not occur in this approach, because the coefficient $2E_r$ plays the role sufficiently. Moreover no Lorentz contraction leads to no transformation, because the variable r of the potential is perpendicular to the motion of the particle.

The form factor and the charge radius are defined and calculated for the wavefunction as

$$F(\boldsymbol{q}^2) = \int \rho(r) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d^3r = Q - \frac{1}{6} q^2 < r^2 > + \cdots\cdots,$$
$$< r^2 > = 4\pi \int \rho(r) r^4 dr = \Gamma(\frac{10}{3})(2a)^{-4/3}, \tag{12}$$

where Q means the electromagnetic charge of the meson and Γ designates the gamma function. Since the relative distance is not the real charge radius of the pion, the density should be replaced to $\rho(r) = Q_{\bar{u}}|\Psi_{\bar{u}}(r_{\bar{u}})|^2 + Q_d|\Psi_d(r_d)|^2$ with $r_u = m_d r/(m_u + m_d)$ and $r_d = m_u r/(m_u + m_d)$. The charge radii for the pion and kaons are calculated as

$$< r^2_{\pi^-} > = \frac{1}{3}\Gamma(\frac{10}{3})\frac{2m_d^2 + m_u^2}{(m_u + m_d)^2}(\frac{9}{32E_r^u k})^{2/3},$$
$$< r^2_{K^-} > = \frac{1}{3}\Gamma(\frac{10}{3})\frac{2m_s^2 + m_u^2}{(m_u + m_s)^2}(\frac{9}{32E_r^u k})^{2/3},$$
$$< r^2_{K^0} > = -\frac{1}{3}\Gamma(\frac{10}{3})\frac{m_s - m_d}{m_s + m_d}(\frac{9}{32E_r^d k})^{2/3}, \tag{13}$$

where the relative energy is calculated as $E_r^1 = (M^2 - m_1^2 - m_2^2)/2(m_1 + m_2)$ as mentioned previously from the energy $E_1' = (M^2 - m_1^2 - m_2^2)/2m_2$ in the rest frame of m_2, and E_1' is calculated from the Lorentz invariance of the momentum squared $(p_1 + p_2)^2_{\text{c.m.}} = (p_1' + p_2')^2_{\text{rest frame of } m_2}$ and the energy calculated below in the center of mass frame. The index 1 of the two particles is assigned to the lighter one through out this letter. Since the number of the unknown parameters is more than that of the equations, more physical processes are needed to calculate. So the decay rates for the above meson are calculated in the following.

From the relation $dN = \rho v d\sigma$, where N is the number of transition particles and v is the relative velocity between the two initial particles, the differential decay rate can be defined by

$$d\Gamma = \frac{S|\Psi(0)|^2}{2E_1 2E_2}|\mathcal{M}|^2(2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2)\frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}, \tag{14}$$

where S is a symmetric factor for the initial particles and $|\Psi(0)|^2 = 4E_r k/3\pi$ from the Eq. (11). The invariant amplitude for the pion decay process $d(p_2) + \bar{u}(p_1) \to \mu^-(k_2) + \bar{\nu}(k_1)$ is calculated as

$$|\mathcal{M}|^2 = 32G_F^2 \cos^2\theta H_{\mu\nu}L^{\mu\nu}, \tag{15}$$

where the hadronic tensor means that $H_{\mu\nu} = \frac{1}{2}Tr[\bar{u}(p_1)\gamma_\mu\frac{1}{2}(1-\gamma_5)u(p_2)\bar{u}(p_2)\gamma_\nu\frac{1}{2}(1-\gamma_5)u(p_1)$ and the leptonic tensor is similar to that for the momentum $-k_1$ and k_2. The integration of the product of the tensors over the whole solid angle is calculated as

$$\int H_{\mu\nu}L^{\mu\nu}d\Omega = \int 4p_1\cdot k_1 p_2\cdot k_2 d\Omega$$
$$= 64\pi E_1E_2(\omega_1\omega_2 + \frac{p^2k^2}{3E_1E_2}), \tag{16}$$

where the energies and momenta are calculated from the relation $q^2 = M^2 = (p_1 + p_2)^2 = (k_1 + k_2)^2$ in the center of mass frame as

$$\boldsymbol{p}^2 = \frac{1}{4M^2}\{(M^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2\},$$
$$E_1 = \frac{1}{2M}(M^2 + m_1^2 - m_2^2),$$
$$E_2 = \frac{1}{2M}(M^2 + m_2^2 - m_1^2),$$
$$\boldsymbol{k}^2 = \frac{1}{4M^2}(M^2 - m_\mu^2)^2,$$
$$\omega_1 = \frac{1}{2M}(M^2 - m_\mu^2),$$
$$\omega_2 = \frac{1}{2M}(M^2 + m_\mu^2), \tag{17}$$

where M is the meson mass which is the bound system of m_1 and m_2. From the above knowledge, the total decay rate is calculated as

$$\Gamma = \frac{G_F^2}{8\pi}|V_{12}|^2 m_\mu^2 M(1-\frac{m_\mu^2}{M^2})^2[\frac{32SME_r^1 k}{3\pi m_\mu^2}\{1+\frac{m_\mu^2}{M^2}+\frac{(1-\frac{m_\mu^2}{M^2})((1-\frac{m_1^2}{M^2}-\frac{m_2^2}{M^2})^2-\frac{4m_1^2m_2^2}{M^4})}{3(1-(\frac{m_1^2}{M^2}-\frac{m_2^2}{M^2})^2)}\}], \tag{18}$$

where the terms in the square bracket correspond to the pion decay constant f_π^2 in the usual calculation for $\pi^- \to \mu^- + \bar{\nu}$ and the kaon decay constant f_K^2 for $K^- \to \mu^- + \bar{\nu}$. From the comparison between the decay and the charge radius of the pion in Refs. [4,5], the space wave function $|\Psi(0)|^2$ in the radius is 19 times larger than that in the decay. The reason is conjectured that the symmetric factor is not properly considered and the form factor of the pion is a dipole type in the reference. As the numerical calculations are shown below, the space wave function in the charge radii agrees to that in the decay rate strikingly, if the symmetric factor S is assumed to be $1/(4\times 9)$. The 1/4 means that the pseudoscalar meson is in the spin signet state, which may have been considered in Refs. [4,5] already. However the color factor 1/9, which means that the decay takes place in the color singlet state and does not occur during the interactions of the exchange of 8 gluons, must have been ignored.

For the typical current quark masses $m_u = 10$ MeV, $m_d = 28.61$ MeV and $m_s = 160$ MeV, the numerical values are calculated and compared with the experimental values [10–13] in the second column of Table I, using the value $k = 34655.5$ MeV^2 from the relation just above Eq. (11) with the estimated value $\alpha_s = 13.35$ in Ref. [1]. The reason for using the same value k through out all the physical quantities in the table is that the mass of the strange quark is known to be so heavy that it can not satisfy the condition $4m_s^2 < q^2$. Instead of the decay rate, the decay constants are regarded as the experimental values for simplicity. The variation of the factor in the brace bracket in Eq. (18) is much small as the variation of the quark mass, actually it is less than 0.1 around its values 1.70 for the pion decay and 1.30 for the kaon decay. Hence the u- and d-quark mass ratio is extracted as 0.35 from the comparison of Eqs. (13) and (18) for the common $E_r^1 k$ in the case of the pion, but the u- and s-quark mass ratio does not give an acceptable value in case of the kaon because of the small value of the kaon decay constant. If the calculated value of the kaon decay constant is replaced to the experimental value, the the u- and s-quark mass ratio is calculated as 0.065 and the CKM matrix element $|V_{us}|$ is estimated as 0.101. This matrix element is out of the range of the present standard values 0.217-0.224 [13,14]. However there may be sufficient room to account for this discrepancy, if the K_{e3} decay $K^+ \to \pi^0 + e^+ + \nu$ is reconsidered carefully with an elegant quark model and the CKM matrix is chosen as the Kobayashi and Maskawa's original parameterization. This numerical calculation gives a new relation $f_\pi/f_K \approx M_\pi/M_K$.

This comparison of the equations can be applied to other models, such as the wavefunction $\Psi(r) = \sqrt{a^3/\pi}e^{-ar}$ for the Coulomb potential $V_{\rm eff} = -\alpha/r$ and $\Psi(r) = (2a/\pi)^{3/4}e^{-ar^2}$ for the harmonic oscillator potential $V_{\rm eff} = br^2$. If the above quark masses are adjusted to the charge radii and taken as the inputs, the decay constants of the pion and kaon are 1.55 and 2.77 MeV for the coulomb potential and 274.2 and 583.9 MeV for the harmonic oscillator potential, respectively. Since the exponential function of the wavefunction reflects the asymptotic behavior of the potential, only the linear potential gives consistent results for the two different physical quantities, that is, charge radii and decay rates.

Finally other current quark masses such as $m_u = 1.5$ MeV [13], $m_d = 4.29$ MeV, and $m_s = 14.77$ MeV give also interesting results as shown in the third column of table I for other suitable value $k = 5057.66$ MeV^2 which means $\alpha_s = 34.0$ in the point of view of Ref. [1] as used previously. These quark masses are adjusted to the new relation $f_\pi/f_K = M_\pi/M_K$. Since the quark mass ratios are reproduced similar to that of the inputs, it is more reliable to adjust the quark masses to the experimental values for the charge radii rather than the decay constants. If the quark masses are adjusted to the decay constants and taken as inputs, then the estimated values for the charge radii of kaons are too large to accept them with ignoring the experimental values. It is also interesting to fit the quark masses to the experimental values for different values k for the pion and kaons, respectively, but it is still impossible to fit the experimental values to the charge radii and the decay constant of the kaons simultaneously with the same current quark masses due to the small kaon decay constant. This is the principal reason that there are four parameter, that is, m_u, m_d, m_s, and k, and five physical quantities, namely, $< r_{\pi^-}^2 >$, $< r_{K^-}^2 >$, $< r_{K^0}^2 >$, f_π, and f_K, but it is difficult to fix the parameters with those physical quantities.

[1] Yongkyu Ko, nucl-th/9901025, (1999).
[2] H. Hershach, Phys. Rev. **D 47**, 3027 (1993).
[3] D. Eyre and J. P. Vary, Phys. Rev. **D 34**, 3467 (1986).
[4] R. Van Royen and V. F. Weisskopf, Nuovo Cimento **50**, 617 (1967).
[5] J. J. J. Kokkedee, *The Quark Model* (W. A. Benjamin, Inc., New York, 1969).
[6] W. Lucha, F. F. Schöberl and D. Gromes, Phys. Rep. **200**, 127 (1991).
[7] T. W. Allen and W. H. Klink, Phys. Rev. **C 58**, 3670 (1998).
[8] A. F. Krutov and V. E. Troitsky, J. Phys. G: Nucl. Part. Phys. **19**, L127 (1993).
[9] H. Kamada and W. Glöckle, Phys. Rev. Lett. **80**, 2547 (1998).
[10] S. R. Amendolia *et al.*, Phys. Lett. **146B**, 116 (1984).
[11] S. R. Amendolia *et al.*, Phys. Lett. **178B**, 435 (1986).
[12] W. R. Molzon, J. Hoffnagle, J. Roehrig, V. L. Telegdi and B. Winstein, Phys. Rev. Lett. **41**, 1213 (1978).
[13] Particle Data Group, C. Caso *et al.*, Eur. Phys. J. **C 3**, 1 (1998).
[14] Particle Data Group, L. Montanet *et al.*, Phys. Rev. **D 50**, 1173 (1994).

physical quantities	calculated values I	calculated values II	experimental values
$< r^2_{\pi^-} >$	0.438 fm^2	0.433 fm^2	0.43 $\pm$ 0.016 fm^2
$< r^2_{K^-} >$	0.347 fm^2	0.227 fm^2	0.34 $\pm$ 0.05 fm^2
$< r^2_{K^0} >$	- 0.145 fm^2	- 0.083 fm^2	- 0.054 $\pm$ 0.026 fm^2
f_π	129.38 MeV	131.15 MeV	130.7 $\pm$ 0.1 $\pm$ 0.36 MeV
f_K	347.52 MeV	463.89 MeV	159.8 $\pm$ 1.4 $\pm$ 0.44 MeV

TABLE I. The charge radius and the decay constant of the pion agree excellently to the experimental values, but those for the kaons do not account for the experimental values properly. The quark masses are adjusted to the radius of K^- in the first calculation, and to the relation $f_\pi/f_K = M_\pi/M_K$, that is, the calculated f_K in the second calculation. The experimental values are taken from Refs. [10-14].